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Coset Space Dimensional Reduction of Einstein-Yang-Mills theory

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ABSTRACT

In the present contribution we extend our previous work by considering the coset space dimensional reduction of higher-dimensional Einstein–Yang–Mills theories including scalar fluctuations as well as Kaluza–Klein excitations of the compactification metric and we describe the gravity-modified rules for the reduction of non-abelian gauge theories.

1 Introduction

In the last four decades we have witnessed a revival of interest in Kaluza–Klein theories, triggered by the realization [1] that non-abelian gauge groups appear naturally when one assumes that the unification takes place in higher dimensions. More specifically, one typically considers a total space-time manifold that can be written as a direct product $M^D = M^4 \times B$, where B is a Riemannian space with a non-abelian isometry group S. The dimensional reduction of this theory leads to gravity coupled to a Yang–Mills theory with a gauge group containing S and scalars in four dimensions. The main advantage of this scenario is the geometrical unification of gravity with the other interactions and the natural emergence of the observed non-abelian gauge symmetries. However, there are problems in the Kaluza-Klein framework.

The most serious obstacle in obtaining a realistic model of the low-energy interactions is that it is impossible to obtain chiral fermions in four dimensions [2]. Fortunately, there is a very interesting resolution to this problem resulting when one adds Yang–Mills fields to the original gravity action. These gauge fields can be responsible for a non-trivial background configuration which could provide chiral fermions to the four-dimensional theory according to the Atiyah-Hizebruch theorem [4]. Moreover, the system admits a stable classical ground state of the required form and the relevant mechanism is known as *spontaneous compactification* [3]. Thus one is led to introduce Yang–Mills fields in higher dimensions. This approach is further justified by other popular unification schemes such as supergravity and heterotic string theory [5].

Gauge fields in the higher-dimensional theory are also welcome from another point of view, since they can provide a potential unification of the low-energy gauge interactions as well as of gauge and Higgs fields. Concerning the latter we should recall that the celebrated Standard Model (SM) of Elementary Particle Physics, which had so far outstanding successes in all its confrontations with experimental results, has also obvious limitations due to the presence of a plethora of free parameters mostly related to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory. The Coset Space Dimensional Reduction (CSDR) [6, 7, 8] was suggesting from the beginning that a unification of the gauge and Higgs sectors can be achieved using higher dimensions. In the CSDR one assumes that the form of space-time is $M^D = M^4 \times S/R$ with S/R being a homogeneous space (obtained as the quotient of the Lie group S by the Lie subgroup R). Then a gauge theory with gauge group G defined on M^D can be dimensionally reduced to M^4 in an elegant way using the symmetries of S/R. In particular, the resulting four-dimensional gauge group is a subgroup of G. The four-dimensional gauge and Higgs fields are simply the surviving components of the gauge fields of the pure higher-dimensional gauge theory.

Similarly, when fermions are introduced [9] the four-dimensional Yukawa and gauge interactions of fermions find also a unified description in the gauge interactions of the higher-dimensional theory. The last step in this unified description in high dimensions is to relate the gauge and fermion fields that have been introduced. A simple way to achieve that is by demanding that the higher-dimensional gauge theory is $\mathcal{N} = 1$ supersymmetric, which requires that the gauge and fermion fields are members of the same vector supermultiplet. A very welcome additional input is that heterotic string theory suggests the dimension and the gauge group of the higher dimensional supersymmetric theory [5]. Moreover, ref. [10] showed that coset spaces with nearly-Kähler geometry yield supersymmetric solutions of heterotic strings in the presence of fluxes and condensates. Therefore, the CSDR might be an appropriate reduction scheme for such compactifications.

The fact that the SM is a chiral theory leads us to consider D-dimensional supersymmetric gauge theories with $D = 4n+2$ [4, 8], which include the ten dimensions suggested by heterotic strings [5]. Concerning supersymmetry, the nature of the four-dimensional theory depends on the nature of the corresponding compact space used to reduce the higher-dimensional theory. Specifically, the reduction over CY spaces leads to supersymmetric theories [5] in four dimensions, the reduction over symmetric coset spaces leads to non-supersymmetric theories, while a reduction over non-symmetric ones leads to softly broken supersymmetric theories [11].

In the present paper, continuing our recent work on the CSDR of the bosonic part of a higher-dimensional Einstein–Yang–Mills theory [12], we apply the CSDR to the gravity sector and describe explicitly the low-energy effective theory. We emphasize that the latter is characterized by a potential for the metric moduli. Furthermore, we revisit the CSDR of gauge theories taking into account the contribution of the dynamical (non-frozen) gravity background and write down the resulting modified constraints and effective action.

2 Geometry of Coset spaces

To describe the geometry of coset spaces we rely on refs. [14, 15]. In the present section we collect the definitions and results that are useful for our discussion. On a coset S/R the Maurer-Cartan 1-form is defined by $e(y) = L^{-1}(y)dL$, where $L(y^a)$ is a coset representative and $a = 1 \ldots dimS/R$. It is the analogue of the left-invariant forms defined on group manifolds and its values are in $Lie(S)$, the Lie algebra of S, i.e. it can be expanded as

$$e(y) = e^A Q_A = e^a Q_a + e^i Q_i, \tag{2.1}$$

where A is a group index, a is a coset index and i is an R-index. e^a is the coframe and e^i is the R-connection. The exterior derivative of the Maurer-Cartan 1-form is

$$de^A = -\frac{1}{2} f^A{}_{BC} e^B \wedge e^C. \tag{2.2}$$

Eq. (2.2) can be expanded as

$$\begin{aligned} de^a &= -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c - f^a{}_{bi} e^b \wedge e^i, \\ de^i &= -\frac{1}{2} f^i{}_{ab} e^a \wedge e^b - \frac{1}{2} f^i{}_{jk} e^j \wedge e^k. \end{aligned} \tag{2.3}$$

The commutation relations obeyed by the generators of S are

$$\begin{aligned} [Q_i, Q_j] &= f_{ij}{}^k Q_k, \\ [Q_i, Q_a] &= f_{ia}{}^b Q_b, \\ [Q_a, Q_b] &= f_{ab}{}^c Q_c + f_{ab}{}^i Q_i. \end{aligned} \tag{2.4}$$

We assume (for reasons analyzed in detail in ref. [14]) that the coset is reductive, i.e. $f_{bi}{}^j = 0$. The normalizer $N(R)$ of R in S is defined as follows

$$N = \{s \in S, \quad sRs^{-1} \subset R\}. \tag{2.5}$$

Since R is normal in $N(R)$ the quotient $N(R)/R$ is a group. The generators Q_a split into two sets $Q_{\hat{a}}, Q_{\bar{a}}$ with $Q_{\hat{a}}$ forming a group which is isomorphic to $N(R)/R$. Then the Lie algebra of S decomposes as

$$S = R + K + L,$$

with

$$[K, K] \subset K, \quad [K, R] = 0, \quad [K, L] \subset L, \quad [L, R] \subset L, \quad [L, L] = L + R. \tag{2.6}$$

Accordingly, the commutation relations (2.4) split as

$$\begin{aligned} [Q_{\hat{a}}, Q_{\hat{b}}] &= f_{\hat{a}\hat{b}}{}^{\hat{c}} Q_{\hat{c}}, \quad [Q_i, Q_{\hat{a}}] = 0, \quad [Q_{\hat{a}}, Q_{\bar{a}}] = f_{\hat{a}\bar{a}}{}^{\bar{b}} Q_{\bar{b}}, \\ [Q_i, Q_{\bar{a}}] &= f_{i\bar{a}}{}^{\bar{b}} Q_{\bar{b}}, \quad [Q_{\bar{a}}, Q_{\bar{b}}] = f_{\bar{a}\bar{b}}{}^{\bar{c}} Q_{\bar{c}} + f_{\bar{a}\bar{b}}{}^i Q_i. \end{aligned} \tag{2.7}$$

Eq. (2.3) is then further decomposed to

$$\begin{aligned} de^{\hat{a}} &= -\frac{1}{2} f^{\hat{a}}{}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}, \\ de^{\bar{a}} &= -\frac{1}{2} f^{\bar{a}}{}_{\bar{b}\bar{c}} e^{\bar{b}} \wedge e^{\bar{c}} - f^{\bar{a}}{}_{\hat{b}\bar{c}} e^{\hat{b}} \wedge e^{\bar{c}} - f^{\bar{a}}{}_{\bar{b}i} e^{\bar{b}} \wedge e^i, \\ de^i &= -\frac{1}{2} f^i{}_{\bar{b}\bar{c}} e^{\bar{b}} \wedge e^{\bar{c}} - \frac{1}{2} f^i{}_{jk} e^j \wedge e^k. \end{aligned} \tag{2.8}$$

An S-invariant metric on S/R is

$$g_{\alpha\beta}(y) = \delta_{ab} e^a_\alpha(y) e^b_\beta(y). \tag{2.9}$$

Using the metric (2.9) the following useful identities can be proved

$$e^a \wedge *_d e^b = \delta^{ab} vol_d, \tag{2.10}$$

$$(e^a \wedge e^b) \wedge *_d (e^c \wedge e^d) = \delta^{ab}_{cd} vol_d, \tag{2.11}$$

$$(e^a \wedge e^b \wedge e^c) \wedge *_d (e^d \wedge e^e \wedge e^f) = \delta^{abc}_{def} vol_d. \tag{2.12}$$

where $*_d$ is the Hodge duality operator on a d-dimensional coset. The Killing vectors associated with the left-isometry group S are

$$K^\alpha_A = D^a_A e^\alpha_a, \tag{2.13}$$

where e^α_a is the inverse vielbein and $D^B_A(s)$ is a matrix in the adjoint representation of S. The coset S/R also posses a right-isometry group which is $N(R)/R$. The relevant Killing vectors are

$$\tilde{K}^\alpha_{\hat{a}} = e^\alpha_{\hat{a}}, \tag{2.14}$$

where $\hat{a} = 1 \ldots dim N(R)/R$ and $e^\alpha_{\hat{a}}$ is the inverse vielbein.

3 The Coset Space Dimensional Reduction

In the present section we present a brief reminder of the Coset Space Dimensional Reduction scheme. The CSDR of a multidimensional gauge field $\hat{A}$ on a coset S/R is a truncation described by a generalized invariance condition

$$\mathcal{L}_{X^I} \hat{A} = DW_I, \tag{3.1}$$

where W_I is a parameter of a gauge transformation associated with the Killing vector X_I of S/R. The relevant invariance condition for the reduction of the metric is

$$\mathcal{L}_{X^I} g_{MN} = 0. \tag{3.2}$$

The generalized invariance condition

$$\mathcal{L}_{X^I} \hat{A} = i_{X^I} d\hat{A} + d i_{X^I} \hat{A} = DW_I = dW_I + [\hat{A}, W_I], \tag{3.3}$$

together with the consistency condition

$$[\mathcal{L}_{X^I}, \mathcal{L}_{X^J}] = \mathcal{L}_{[X^I, X^J]}, \tag{3.4}$$

impose constraints on the gauge field. The detailed analysis of the constraints (3.3) and (3.4), given in refs.[7, 8], provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction.

Instead, we may use the following ansatz for the gauge fields, which was shown in [12] to be equivalent to the CSDR ansatz and it is similar to the Scherk-Schwartz reduction ansatz:

$$\hat{A}^{\tilde{I}}(x,y) = A^{\tilde{I}}(x) + \chi^{\tilde{I}}_{\alpha}(x,y)dy^{\alpha}, \tag{3.5}$$

where

$$\chi^{\tilde{I}}_{\alpha}(x,y) = \phi^{\tilde{I}}_{A}(x)e^{A}_{\alpha}(y). \tag{3.6}$$

The objects $\phi_A(x)$, which take values in the Lie algebra of G, are coordinate scalars in four dimensions and they can be identified with Higgs fields.

4 Gravity and CSDR

Usually one studies higher-dimensional gauge theories and constructs four-dimensional unified models, in a frozen gravity background, i.e., the internal metric is of the form (2.9). In this section we search for gravity backgrounds consistent with CSDR in the sense of eq. (3.2) but including fluctuations of the metric [17, 18, 19, 20]. We begin by examining a D-dimensional Einstein–Yang–Mills Lagrangian

$$\mathcal{L} = \hat{R} *_D \mathbf{1} - \frac{1}{2}Tr\hat{F}_{(2)} \wedge *_D\hat{F}_{(2)} - \hat{\lambda}_{(D)} *_D \mathbf{1}, \tag{4.1}$$

where $\hat{F}_{(2)} = d\hat{A}_{(1)} + \hat{A}_{(1)} \wedge \hat{A}_{(1)}$ is a gauge field with values in the Lie algebra of a group G, $\hat{R}$ is the curvature scalar and $\hat{\lambda}_{(D)}$ is the cosmological constant in D-dimensions. A general ansatz for the metric is

$$d\hat{s}^2_{(D)} = ds^2_{(4)} + h_{\alpha\beta}(x,y)(dy^{\alpha} - \mathcal{A}^{\alpha}(x,y))(dy^{\beta} - \mathcal{A}^{\beta}(x,y)), \tag{4.2}$$

where A^{α} is the Kaluza-Klein gauge field

$$\mathcal{A}^{\alpha}(x,y) = \mathcal{A}^{I}(x)K^{\alpha}_{(I)}(y), \quad \mathcal{A}^{I}(x) = \mathcal{A}^{I}_{\mu}(x)dx^{\mu}, \tag{4.3}$$

and $K_{(I)}(y) = K^{\alpha}_{(I)}(y)\frac{\partial}{\partial y^{\alpha}}$ are at most the $dimS + dim(N(R)/R)$ Killing vectors of the coset S/R or an appropriate subset. A well known problem with coset reductions is that we cannot consistently allow Kaluza–Klein gauge fields from the full isometry group S of the coset S/R to survive.

According to refs [21], [16] the correct ansatz leading to a consistent truncation of the theory is to consider Kaluza-Klein gauge fields belonging to the $N(R)/R$ part of the isometry group S/R

$$\mathcal{A}^{\alpha}(x,y) = \mathcal{A}^{\hat{a}}(x)\tilde{K}^{\alpha}{}_{\hat{a}}(y). \tag{4.4}$$

Now in the ansatz (4.2) we have

$$\eta^{\hat{a}} = e^{\hat{a}}_{\alpha}(dy^{\alpha} - A^{\hat{b}}(x)\tilde{K}^{\alpha}{}_{\hat{b}}(y)) = e^{\hat{a}} - A^{\hat{a}}(x), \tag{4.5}$$

given that

$$e^{\hat{a}}_{\alpha}\tilde{K}^{\alpha}_{\hat{b}} = \delta^{\hat{a}}_{\hat{b}}, \tag{4.6}$$

with $\tilde{K}^{\alpha}_{\hat{b}}$ being the Killing vectors of the right isometries $N(R)/R$. The rest of the 1-forms are

$$\eta^{\bar{a}} = e^{\bar{a}}, \quad e^{i} = e^{i}_{a}e^{a}. \tag{4.7}$$

For $\eta^{\hat{a}}$ we find that

$$D\eta^{\hat{a}} \equiv d\eta^{\hat{a}} + f^{\hat{a}}{}_{\hat{b}\hat{c}}\mathcal{A}^{\hat{b}} \wedge \eta^{\hat{c}} = -\mathcal{F}^{\hat{a}} - \frac{1}{2}f^{\hat{a}}{}_{\hat{b}\hat{c}}\eta^{\hat{b}} \wedge \eta^{\hat{c}}, \tag{4.8}$$

where $\mathcal{F}^{\hat{b}}$ is the field strength of the Kaluza–Klein gauge field $\mathcal{A}^{\hat{b}}$ defined by

$$\mathcal{F}^{\hat{b}} = d\mathcal{A}^{\hat{b}} + \frac{1}{2}f^{\hat{b}}{}_{\hat{c}\hat{d}}\mathcal{A}^{\hat{c}} \wedge \mathcal{A}^{\hat{d}}. \tag{4.9}$$

Now the metric ansatz for a general S-invariant metric takes the form

$$d\hat{s}^{2}_{(D)} = e^{2\alpha\phi(x)}\eta_{mn}e^{m}e^{n} + e^{2\beta\phi(x)}\gamma_{ab}(x)\eta^{a}\eta^{b}, \tag{4.10}$$

from which we read the vielbeins (the notation is close to that one used in ref. [13]):

$$\hat{e}^{m} = e^{\alpha\phi}e^{m}, \quad \hat{e}^{a} = e^{\beta\phi}\Phi^{a}_{b}(x)\eta^{b}, \tag{4.11}$$

with

$$\gamma_{cd}(x) = \delta_{ab}\Phi^{a}_{c}(x)\Phi^{b}_{d}(x). \tag{4.12}$$

Φ is a matrix of unit determinant so there exists a set $(\Phi^{-1})^{b}_{a}$ of fields satisfying

$$(\Phi^{-1})^{c}_{a}(\Phi^{-1})^{d}_{b}\gamma_{cd} = \delta_{ab}. \tag{4.13}$$

Next we calculate the exterior derivatives of the vielbeins

$$d\hat{e}^{m} = -\omega^{m}{}_{n} \wedge \hat{e}^{n} - \alpha e^{-\alpha\phi}\partial_{\nu}\phi\hat{e}^{m} \wedge \hat{e}^{n}, \tag{4.14}$$

$$d\hat{e}^a = -\tilde{f}^a{}_{ib} e^i \wedge \hat{e}^b + e^{-\alpha\phi} D_{bn}{}^a \hat{e}^{nb} + \beta e^{-\alpha\phi} \partial_m \phi \hat{e}^{ma} - \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_{mn} \hat{e}^{mn} - \frac{1}{2} \tilde{f}^a{}_{bc} \hat{e}^{bc}, \tag{4.15}$$

where

$$e^{ab} \equiv e^a \wedge e^b, \qquad \mathcal{F}^a_{mn} \equiv \Phi^a_{\hat{a}} \mathcal{F}^{\hat{a}}_{mn}, \tag{4.16}$$

and

$$\begin{aligned} &\tilde{f}^a{}_{ib} = \Phi^a_c (\Phi^{-1})^d_b f^c{}_{id}, \quad D_{bn}{}^a = (\Phi^{-1})_b{}^c D_n \Phi^a{}_c, \\ &\tilde{f}^a{}_{bc} = \Phi^a{}_d (\Phi^{-1})_b{}^e (\Phi^{-1})_c{}^f f^d{}_{ef}. \end{aligned} \tag{4.17}$$

Subsequently we compute the spin connections

$$\hat{\omega}_{mn} = \omega_{mn} + \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_{mn} \hat{e}^a + \alpha e^{-\alpha\phi} (\partial_n \phi \eta_{ml} \hat{e}^l - \partial_m \phi \eta_{nl} \hat{e}^l), \tag{4.18}$$

$$\hat{\omega}_{ma} = -e^{-\alpha\phi} P_{mab} \hat{e}^b - \beta e^{-\alpha\phi} \partial_m \phi \hat{e}^a + \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}_{aml} \hat{e}^l, \tag{4.19}$$

$$\hat{\omega}_{ab} = -\tilde{f}_{iab} e^i + e^{-\alpha\phi} Q_{mab} \hat{e}^m + e^{-\beta\phi} \tilde{C}_{cab} \hat{e}^c, \tag{4.20}$$

where

$$\begin{aligned} \tilde{C}_{cab} &= \frac{1}{2} (\tilde{f}^c{}_{ab} + \tilde{f}^b{}_{ac} - \tilde{f}^a{}_{bc}), \\ P_{mab} &= \frac{1}{2} [(\Phi^{-1})^c_a D_m \Phi^b_c + (\Phi^{-1})^c_b D_m \Phi^a_c], \\ Q_{mab} &= \frac{1}{2} [(\Phi^{-1})^c_a D_m \Phi^b_c - (\Phi^{-1})^c_b D_m \Phi^a_c], \\ D_m \Phi^a_d &= \partial_m \Phi^a_d - f^c{}_{d\hat{b}} \mathcal{A}^{\hat{b}}_m \Phi^a_c. \end{aligned} \tag{4.21}$$

It is well-known that the curvature scalar of the gravitational Lagrangian can be written as

$$\hat{R} *_D \mathbf{1} = \hat{\Theta}_{AB} \wedge *_D (\hat{e}^a \wedge \hat{e}^B), \tag{4.22}$$

where $A = m, a$ and Θ_{AB} are the curvature 2-forms calculated from eqs. (4.18), (4.19) and (4.20). Then the Lagrangian is reduced to four dimensions provided we impose the following constraints

$$\begin{aligned} &-\frac{1}{2} \tilde{f}^a_{ib} f^i_{jk} + \tilde{f}^a_{jc} \tilde{f}^c_{kb} = 0, \\ &-\tilde{C}^a_{cb} \tilde{f}^c_{id} + \tilde{C}^a_{dc} \tilde{f}^c_{ib} - \tilde{C}^c_{db} \tilde{f}^a_{ic} = 0. \end{aligned} \tag{4.23}$$

The constraints (4.23) can be shown to be satisfied using the Jacobi identities and the invariance of the metric.

Finally, we can write down the reduced Lagrangian in the form

$$\begin{aligned}
\mathcal{L} &= e^{4\alpha+d\beta}\Big(e^{-2\alpha\phi}R*\mathbf{1} - e^{-2\alpha\phi}*P_{ab}\wedge P_{ab} - \frac{1}{2}e^{2(\beta-2\alpha)\phi}\gamma_{ab}*\mathcal{F}^a\wedge\mathcal{F}^b \\
&+ e^{-2\alpha\phi}((3\alpha+d\beta)^2-(3\alpha^2+d\beta^2))*d\phi\wedge d\phi \\
&- \frac{1}{4}e^{-2\beta\phi}(\gamma_{ab}\gamma^{cd}\gamma^{ef}f^a_{\ ce}f^b_{\ df}+2\gamma^{ab}f^c_{\ da}f^d_{\ cb}+4e^{2\beta\phi}\gamma^{ab}f_{iac}f^{ic}_b)*\mathbf{1}+\lambda_D*\mathbf{1}\Big).
\end{aligned} \tag{4.24}$$

In order to obtain the correct kinetic terms in four dimensions we should choose

$$\alpha = -\sqrt{\frac{d}{4d+8}}, \qquad \beta = -\frac{2\alpha}{d}. \tag{4.25}$$

To the set of the imposed constraints we should add that the condition that Φ is a matrix of unit determinant and that the structure constants of S are traceless and fully antisymmetric. The final form of the reduced Lagrangian is

$$\mathcal{L} = R*\mathbf{1} - *P_{ab}\wedge P_{ab} - \frac{1}{2}e^{2(\beta-\alpha)\phi}\gamma_{\hat{a}\hat{b}}*\mathcal{F}^{\hat{a}}\wedge\mathcal{F}^{\hat{b}} - \frac{1}{2}*d\phi\wedge d\phi - V(\phi), \tag{4.26}$$

where the potential for the metric moduli fields reads

$$V = \frac{1}{4}e^{2(\alpha-\beta)\phi}(\gamma_{ab}\gamma^{cd}\gamma^{ef}f^a_{\ ce}f^b_{\ df}+2\gamma^{ab}f^c_{\ da}f^d_{\ cb}+4e^{2\beta\phi}\gamma^{ab}f_{iac}f^{ic}_b-4e^{2\beta\phi}\lambda_D)*\mathbf{1}. \tag{4.27}$$

Note that the first two terms in eq. (4.27) have a non-zero contribution only in the case of non-symmetric coset spaces.

5 Reduction of the Gauge Sector: Gravity Modification of the CSDR Rules

In this section we reduce the Yang–Mills Lagrangian in the presence of fluctuating gravity. The ansatz for the higher dimensional gauge field is

$$\hat{A}^{\tilde{I}} = A^{\tilde{I}} + \phi^{\tilde{I}}_A\eta^A, \tag{5.1}$$

where

$$\eta^{\hat{a}} = e^{\hat{a}} - \mathcal{A}^{\hat{a}}, \quad \eta^{\bar{a}} = e^{\bar{a}}, \quad \eta^i = e^i = e^i_a e^a,$$

and $\tilde{I}$ is a gauge group index. Calculating the field strength

$$\hat{F} = \hat{d}\hat{A}^{\tilde{I}} + \frac{1}{2}f^{\tilde{I}}_{\ \tilde{J}\tilde{K}}\hat{A}^{\tilde{J}}\wedge\hat{A}^{\tilde{K}}, \tag{5.2}$$

we find

$$\hat{F}^{\tilde{I}} = (F^{\tilde{I}} - \mathcal{F}^{\hat{a}}\phi^{\tilde{I}}_{\hat{a}}) + D\phi^{\tilde{I}}_A\wedge\eta^A - \frac{1}{2}F^{\tilde{I}}_{AB}\eta^A\wedge\eta^B, \tag{5.3}$$

where $\mathcal{F}^{\hat{a}}$ is the KK gauge field and

$$F^{\tilde{I}} = dA^{\tilde{I}} + \frac{1}{2} f^{\tilde{I}}{}_{\tilde{J}\tilde{K}} A^{\tilde{J}} \wedge A^{\tilde{K}}, \tag{5.4}$$

with

$$F^{\tilde{I}}_{AB} = f^{C}{}_{AB} \phi^{\tilde{I}}_{C} - f^{\tilde{I}}{}_{\tilde{J}\tilde{K}} \phi^{\tilde{J}}_{A} \phi^{\tilde{K}}_{B}, \tag{5.5}$$

and

$$D\phi^{\tilde{I}}_{A} = d\phi^{\tilde{I}}_{A} + f^{C}{}_{AB} \mathcal{A}^{B} \phi^{\tilde{I}}_{C} + f^{\tilde{I}}{}_{\tilde{J}\tilde{K}} A^{J} \phi^{\tilde{K}}_{A}. \tag{5.6}$$

To reduce the higher dimensional Yang–Mills Lagrangian we dualize eq. (5.3) to

$$*_D \hat{F}^{\tilde{I}} = *_4 (F^{\tilde{I}} - \mathcal{F}^{\hat{a}} \phi^{\tilde{I}}_{\hat{a}}) \wedge vol_d + e^{\alpha\phi - \beta\phi} *_4 D\phi^{\tilde{I}}_{A} \wedge *_d \tilde{\eta}^{A} - \frac{1}{2} e^{2\alpha\phi - 2\beta\phi} F^{\tilde{I}}_{AB} vol_4 \wedge *_d (\tilde{\eta}^{A} \wedge \tilde{\eta}^{B}), \tag{5.7}$$

and insert everything in

$$\mathcal{L} = -\frac{1}{2} Tr \hat{F} \wedge *_D \hat{F}.$$

The result is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} e^{-2\alpha\phi} (F^{\tilde{I}} - \mathcal{F}^{\hat{a}} \phi^{\tilde{I}}_{\hat{a}}) \wedge *_4 (F^{\tilde{I}} - \mathcal{F}^{\hat{a}} \phi^{\tilde{I}}_{\hat{a}}) \wedge vol_d - \frac{1}{2} e^{-2\beta\phi} D\phi^{\tilde{I}}_{A} \wedge *_4 D\phi^{\tilde{I}}_{B} \wedge \tilde{\eta}^{A} \wedge *_d \tilde{\eta}^{B} \\ &+ \frac{1}{4} e^{2\alpha\phi - 4\beta\phi} F_{AB} F_{CD} vol_4 \wedge \tilde{\eta}^{A} \wedge \tilde{\eta}^{B} \wedge *_d (\tilde{\eta}^{C} \wedge \tilde{\eta}^{D}), \end{aligned} \tag{5.8}$$

where

$$\tilde{\eta}^{a} = (\Phi^{-1})^{a}_{b} \eta^{b}, \quad \tilde{\eta}^{i} = e^{i}_{a} (\Phi^{-1})^{a}_{b} \eta^{b}. \tag{5.9}$$

To reduce eq. (5.8) we must impose the constraints

$$D\phi^{\tilde{I}}_{i} = 0, \quad F^{\tilde{I}}_{ij} = F^{\tilde{I}}_{aj} = 0, \tag{5.10}$$

and

$$D\phi_{\hat{a}} = F_{\hat{a}\hat{b}} = F_{\hat{a}\bar{b}} = F_{\hat{a}i} = 0. \tag{5.11}$$

The first set is the usual CSDR constraints described in detail at various places (e.g.[12]). We concentrate on the gravity-induced second set. From the condition

$$F^{\tilde{I}}_{\hat{a}\hat{b}} = f^{\hat{c}}{}_{\hat{a}\hat{b}} \phi^{\tilde{I}}_{\hat{c}} - [\phi_{\hat{a}}, \phi_{\hat{b}}]^{\tilde{I}} = 0,$$

we conclude that $\phi_{\hat{a}}$ are the generators of an $N(R)/R$ subgroup of H (remember that R has no $N(R)/R$ subgroup and H is the centralizer of the embedding of R on G, the higher dimensional gauge group). We conclude also that

$$f^{\hat{c}}{}_{\hat{a}\hat{b}} \phi^{\tilde{I}}_{\hat{c}} = f^{\tilde{I}}{}_{\tilde{J}\tilde{K}} \phi^{\tilde{J}}_{\hat{a}} \phi^{\tilde{K}}_{\hat{b}}. \tag{5.12}$$

Given the condition (5.12) the constraint $D\phi^{\tilde{I}}_{\hat{a}} = 0$ yields ($\phi_{\hat{a}}$ is constant)

$$f^{\tilde{I}}{}_{\tilde{J}\tilde{K}}\phi^{\tilde{J}}_{\hat{a}}\phi^{\tilde{K}}_{\hat{b}}\mathcal{A}^{\hat{b}} + f^{\tilde{I}}{}_{\tilde{J}\tilde{K}}A^{\tilde{J}}\phi^{\tilde{K}}_{\hat{b}} = 0. \tag{5.13}$$

Eq. (5.13) determines the gauge field belonging to the $N(R)/R$ part of H in terms of the Kaluza–Klein gauge fields

$$A^{\tilde{I}} = \mathcal{A}^{\hat{b}}\phi^{\tilde{I}}_{\hat{b}}. \tag{5.14}$$

Calculating the corresponding field strength we find

$$F^{\tilde{I}} = d\mathcal{A}^{\hat{a}}\phi^{\tilde{I}}_{\hat{a}} + \frac{1}{2}f^{\hat{c}}{}_{\hat{a}\hat{b}}\mathcal{A}^{\hat{a}} \wedge \mathcal{A}^{\hat{b}}\phi^{\tilde{I}}_{\hat{c}} = \mathcal{F}^{\hat{a}}\phi^{\tilde{I}}_{\hat{a}}. \tag{5.15}$$

This is exactly the term subtracted from $F^{\tilde{I}}$ in eq. (5.3), thus leaving a surviving gauge group K obtained from the decompositions

$$G \supset R \times H$$

and

$$H \supset (N(R)/R) \times K.$$

The constraint

$$F_{\hat{a}i} = [\phi_{\hat{a}}, \phi_i] = 0$$

is satisfied trivially while the representations in which the scalars $\phi_{\hat{a}}$ belong are determined by

$$\begin{aligned} F_{\hat{a}\bar{b}} &= f^{\bar{c}}{}_{\hat{a}\bar{b}}\phi_{\bar{c}} - [\phi_{\hat{a}}, \phi_{\bar{b}}] = 0, &(5.16)\\ F_{\bar{a}i} &= f^{\hat{c}}{}_{\bar{a}i}\phi_{\hat{c}} - [\phi_{\bar{a}}, \phi_i] = 0. &(5.17)\end{aligned}$$

These constraints are solved by considering the following decompositions of S and G

$$\begin{aligned} S &\supset R \times (N(R)/R), \\ adS &= adR + adN(R)/R + \sum(r_i, n_i) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} G &\supset R \times (N(R)/R) \times K, \\ adG &= (adR, 1, 1) + (1, adN(R)/R, 1) + (1, 1, adK) + \sum(l_i, m_i, k_i). \end{aligned} \tag{5.19}$$

As in the pure Yang–Mills case there is a k_i multiplet of scalar fields surviving when $(r_i, n_i) = (l_i, m_i)$.

Collecting the various terms we obtain the Lagrangian

$$\mathcal{L} = -\frac{1}{2}e^{-2\alpha\phi}F^{\tilde{I}}\wedge *_4 F^{\tilde{I}}\wedge vol_d - \frac{1}{2}e^{-2\beta\phi}\gamma^{\bar{a}\bar{b}}D\phi^{\tilde{I}}_{\bar{a}}\wedge *_4 D\phi^{\tilde{I}}_{\bar{b}}\wedge vol_d + \frac{1}{4}e^{2\alpha\phi-4\beta\phi}\gamma^{\bar{a}\bar{c}}\gamma^{\bar{b}\bar{d}}F_{\bar{a}\bar{b}}F_{\bar{c}\bar{d}}vol_4\wedge vol_d, \tag{5.20}$$

with gauge group K and scalars in specific representations of K subject to the potential

$$V_{gt} = -\frac{1}{4}e^{(2\alpha\phi-4\beta\phi)}\gamma^{\bar{a}\bar{c}}\gamma^{\bar{b}\bar{d}}F_{\bar{a}\bar{b}}F_{\bar{c}\bar{d}}. \tag{5.21}$$

6 Conclusions

We have studied higher-dimensional Einstein–Yang–Mills theories and examined their Coset Space Dimensional Reduction using an approach similar to that of ref. [13] and combined with the method of Coset Space Dimensional Reduction of gauge theories introduced in ref. [7]. We found that the expected four-dimensional gauge theory coming from CSDR considerations with frozen metric is indeed enhanced by the Kaluza–Klein modes of the metric. However, the emergence of the full isometry of the coset as a part of the four-dimensional gauge group is not permitted. In addition, we showed how the four-dimensional potential is modified from the new scalar fields in the case of non-symmetric coset spaces.

Ref. [10] uncovered supersymmetric vacua of heterotic supergravity (with fluxes and condensates) of the form $M_{1,3}\times S/R$, with S/R being a homogeneous nearly-Kähler manifold. It would be interesting to perform explicitly the reduction on these manifolds using the scheme developed in this work and compare it with the approach of [24] for reduction on $SU(3)$ structure manifolds.

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